Phys 410
Fall 2014
Lecture #13 Summary
14 October, 2014

We discussed several examples of constrained systems by the Lagrangian method. The key step is to identify the number of degrees of freedom in the problem and find the most efficient set of generalized coordinates. Examining the constraints in the system is often a good way to identify the appropriate generalized coordinates. Writing down the kinetic and potential energies in terms of these generalized coordinates is often facilitated by using Cartesian or cylindrical or spherical coordinates, and then converting completely to the generalized coordinates.

We did the example of the Atwood machine for a frictionless and inertia-less pulley supporting two different masses. The masses can each move in one dimension (which we called x and y), and their motion is constrained because they are on either end of a string of fixed length. The constraint is that the string length is $\ell = x + y + \pi R$, where R is the radius of the pulley. With this constraint incorporated, the Lagrangian can be written as $\mathcal{L}(x, \dot{x}) = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + const.$ Note that the constant plays no role in the dynamics since it disappears when both of the derivatives $(\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial \dot{x}})$ are taken. The resulting equation of motion is $\ddot{x} = g\frac{m_1 - m_2}{m_1 + m_2}$. Again note that the constraining force (the tension in the string) was never mentioned or considered in the process. The tension is essential to the traditional Newton's second law approach to solving this problem.

The next example was the problem of a frictionless block sliding down the side of a wedge of angle α which is sliding horizontally over a frictionless surface. Because the block and wedge are constrained to remain in contact, and the wedge and horizontal surface are also constrained to remain in contact, there are really only two degrees of freedom in this problem: the displacement of the wedge in the horizontal direction (q_2), and the displacement of the block down the wedge (q_1). The kinetic and potential energies can be written in terms of these coordinates and their time derivatives. We found that the horizontal component of momentum is conserved, and that the block moves down the wedge with a constant acceleration that depends of the mass of the block and wedge, as well as the angle α. The time for the block to reach the bottom of the wedge is just that of a particle moving with constant acceleration.

The rotating bead on a loop problem was then analyzed. A bead of mass m is constrained to move on a vertical circular loop of radius R, and the loop is set into rotation about the vertical axis through the loop center, at angular frequency ω. There is a single generalized coordinate θ, which is the angle that the bead makes with respect to the vertically-down direction from the center of the loop. There are two components of velocity for the bead, one around the loop ($v_\theta = R\dot{\theta}$) and the other around the vertical axis ($v_\varphi = \rho\omega = R\sin\theta\,\omega$). The Lagrangian is $\mathcal{L}(\theta,\dot{\theta}) = T - U = \frac{mR^2}{2}\left(\dot{\theta}^2 + \omega^2 \sin\theta^2\right) - mgR(1-\cos\theta)$. The resulting equation of motion is $\ddot{\theta} = (\omega^2\cos\theta - g/R)\sin\theta$. This cannot be solved in closed form for $\theta(t)$. Note that the equation reduces to the equation of motion of a pendulum in the limit $\omega \to 0$.

Even though we cannot solve this equation for $\theta(t)$, we can learn much about the possible equilibrium solutions to the equation. From the in-class demonstration we showed that there are several different equilibrium points for the bead while the loop is rotating. The equilibrium points are those special angles θ_0 where a particle can be placed with no initial velocity $\dot{\theta} = 0$ and will stay there because the acceleration is zero, $\ddot{\theta} = 0$. The zeroes of the above equation of motion come from the two terms in the product on the RHS. The first are those for which $\sin\theta_0 = 0$, which include $\theta_0 = 0, \pi$. The position $\theta_0 = \pi$ is always unstable, while that for $\theta_0 = 0$ is stable for low angular velocities ω. The other equilibrium points are given by the zero of the term in parentheses: $\cos\theta_0 = g/\omega^2 R$. However, since the magnitude of $\cos\theta_0$ is bounded, this requires a certain minimum angular velocity, or greater, to be satisfied: $\omega \geq \sqrt{g/R}$. There are two equilibrium angles in this case: $\theta_0 = \pm\cos^{-1}(g/\omega^2 R)$, both of which are stable when they exist. In summary, the angle $\theta_0 = 0$ is stable for $\omega < \sqrt{g/R}$, and it bifurcates (becomes unstable) into two other stable points at $\theta_0 = \pm\cos^{-1} g/\omega^2 R$. In the limit as $\omega \to \infty$, the angles become $\theta_0 = \pm\pi/2$, which is the 'outside' of the circular hoop.

The final example of this lecture is a block sliding down a slope with friction. We use this example to demonstrate the weakness of the Lagrangian mechanics and learn how to make it work again by adding additional terms in the Lagrange equation. For a block with mass m sliding down a slope with angle α, if the friction force is F_r, we have solve this example with Newtonian mechanics, and the equation of motion is $\ddot{q} = g\sin\alpha - \frac{F_r}{m}$, where q is the generalized coordinate along the slope. If we use the Lagrangian $\mathcal{L} = T - U$, we cannot include the friction force, a non-conservative force. The method to solve this problem is to add the friction force as a new term in the generalized force in the Lagrange equation as $\frac{\partial\mathcal{L}}{\partial q} - F_r = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}}$. This modified Lagrange equation will give the correct equation of motion.

For more information of the Lagrangian mechanics with non-conservative forces, please refer to Section 6.5 of this article:

http://ice.as.arizona.edu/~dpsaltis/Phys422/chapter6.pdf